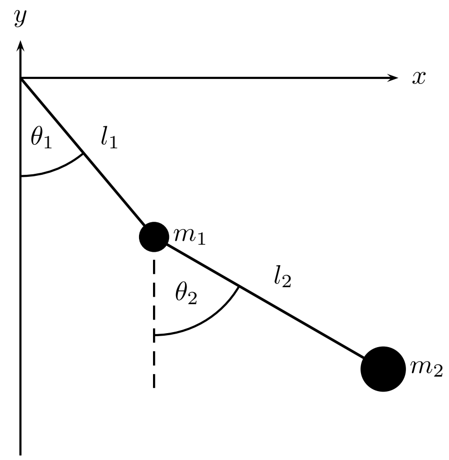
1 Pendule double



$$
\begin{aligned}
x_1 &= l_1 \sin\theta_1 \\
y_1 &= -l_1 \cos\theta_1 \\
x_2 &= l_1 \sin\theta_1 + l_2 \sin\theta_2 \\
y_2 &= -l_1 \cos\theta_1 - l_2 \cos\theta_2
\end{aligned}
$$

The derivatives

$$
\begin{aligned}
\dot{x}_1 &= l_1\dot{\theta}_1 \cos\theta_1 \\
\dot{y}_1 &= l_1\dot{\theta}_1 \sin\theta_1 \\
\dot{x}_2 &= l_1\dot{\theta}_1 \cos\theta_1 + l_2\dot{\theta}_2 \cos\theta_2 \\
\dot{y}_2 &= l_1\dot{\theta}_1 \sin\theta_1 + l_2\dot{\theta}_2 \sin\theta_2
\end{aligned}
$$

The squares of the derivatives

$$
\begin{aligned}
\dot{x}_1^2 + \dot{y}_1^2 &= l_1^2\dot{\theta}_1^2 \cos^2\theta_1 + l_1^2\dot{\theta}_2^2 \sin^2\theta_1 = l_1^2\dot{\theta}_1^2 \\
\dot{x}_2^2 + \dot{y}_2^2 &= \Big(l_1\dot{\theta}_1 \cos\theta_1 + l_2\dot{\theta}_2 \cos\theta_2\Big)^2 + \Big(l_1\dot{\theta}_1 \sin\theta_1 + l_2\dot{\theta}_2 \sin\theta_2\Big)^2 \\
&= \quad l_1^2\dot{\theta}_1^2 \cos^2\theta_1 + 2l_1\dot{\theta}_1 \cos\theta_1 l_2\dot{\theta}_2 \cos\theta_2 + l_2^2\dot{\theta}_2^2 \cos^2\theta_2 + \\
&\quad + l_1^2\dot{\theta}_1^2 \sin^2\theta_1 + 2l_1\dot{\theta}_1 \sin\theta_1 l_2\dot{\theta}_2 \sin\theta_2 + l_2^2\dot{\theta}_2^2 \sin^2\theta_2 \\
&= l_1^2\dot{\theta}_1^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) + l_2^2\dot{\theta}_2^2
\end{aligned}
$$

The kinetic energy

$$
\begin{aligned}
T &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \\
&= \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)\right) \\
&= \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)
\end{aligned}
$$

Potential energy

$$
\begin{aligned}
U &= m_1 g y_1 + m_2 g y_2 \\
&= -(m_1 + m_2)g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2
\end{aligned}
$$

The Lagrange function

$$
\begin{aligned}
L &= T - U \\
&= \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1 + m_2)g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2
\end{aligned}
$$

For θ_1, $\dot\theta_1$, we get

$$\begin{aligned}
\frac{\partial L}{\partial \theta_1} &= -m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2) - (m_1+m_2) g l_1 \sin\theta_1 \\
\frac{\partial L}{\partial \dot\theta_1} &= (m_1+m_2) l_1^2 \dot\theta_1 + m_2 l_1 l_2 \dot\theta_2 \cos(\theta_1-\theta_2) \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\theta_1} &= (m_1+m_2) l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot\theta_2 (\dot\theta_1 - \dot\theta_2)\sin(\theta_1-\theta_2)
\end{aligned}$$

Thus

$$(m_1+m_2) l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1-\theta_2) + m_2 l_1 l_2 \dot\theta_2^2 \sin(\theta_1-\theta_2) + (m_1+m_2) g l_1 \sin\theta_1 = 0$$

Division by $(m_1+m_2)l_1$

$$l_1\ddot\theta_1 + \frac{m_2}{m_1+m_2} l_2 \cos(\theta_1-\theta_2)\ddot\theta_2 + \frac{m_2}{m_1+m_2} l_2 \dot\theta_2^2 \sin(\theta_1-\theta_2) + g\sin\theta_1 = 0$$

For θ_2, $\dot\theta_2$, we get

$$\begin{aligned}
\frac{\partial L}{\partial \theta_2} &= -m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1-\theta_2) - m_2 g l_2 \sin\theta_2 \\
\frac{\partial L}{\partial \dot\theta_2} &= m_2 l_2^2 \dot\theta_2 + m_2 l_1 l_2 \dot\theta_1 \cos(\theta_1-\theta_2) \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\theta_2} &= m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot\theta_1 (\dot\theta_1 - \dot\theta_2)\sin(\theta_1-\theta_2)
\end{aligned}$$

Thus

$$m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot\theta_1^2 \sin(\theta_1-\theta_2) + m_2 g l_2 \sin\theta_2 = 0$$

Division by $m_2 l_2$

$$l_1\cos(\theta_1-\theta_2)\ddot\theta_1 + l_2\ddot\theta_2 - l_1\dot\theta_1^2\sin(\theta_1-\theta_2) + g\sin\theta_2 = 0$$

2 ODEs in normal form, amenable to Runge-Kutta

The system of differential equations is:

$$\begin{aligned}
l_1\ddot\theta_1 + \frac{m_2}{m_1+m_2} l_2 \cos(\theta_1-\theta_2)\ddot\theta_2 + \frac{m_2}{m_1+m_2} l_2 \dot\theta_2^2 \sin(\theta_1-\theta_2) + g\sin\theta_1 = 0 \\
l_1\cos(\theta_1-\theta_2)\ddot\theta_1 + l_2\ddot\theta_2 - l_1\dot\theta_1^2\sin(\theta_1-\theta_2) + g\sin\theta_2 = 0
\end{aligned}$$

With $\mu = \dfrac{m_2}{m_1+m_2}$ the system simplifies to

$$l_1\ddot\theta_1 + \mu l_2 \cos(\theta_1-\theta_2)\ddot\theta_2 + \mu l_2 \dot\theta_2^2 \sin(\theta_1-\theta_2) + g\sin\theta_1 = 0 \tag{1}$$

$$l_1\cos(\theta_1-\theta_2)\ddot\theta_1 + l_2\ddot\theta_2 - l_1\dot\theta_1^2\sin(\theta_1-\theta_2) + g\sin\theta_2 = 0 \tag{2}$$

Eliminating $\ddot\theta_2$ and solving for $\ddot\theta_1$: $(1) - \mu\cos(\theta_1-\theta_2)\cdot(2)$ gives

$$l_1\left(1-\mu\cos^2(\theta_1-\theta_2)\right)\ddot\theta_1 + \mu l_2 \dot\theta_2^2\sin(\theta_1-\theta_2) + g\sin\theta_1 + \frac{1}{2}\mu l_1 \sin(2(\theta_1-\theta_2))\dot\theta_1^2 - \mu g \sin\theta_2\cos(\theta_1-\theta_2) = 0$$

With $\lambda = \dfrac{l_1}{l_2}$ we simplify again

$$\left(1-\mu\cos^2(\theta_1-\theta_2)\right)\ddot\theta_1 + \frac{\mu}{\lambda}\dot\theta_2^2\sin(\theta_1-\theta_2) + \frac{g}{l_1}\sin\theta_1 + \frac{1}{2}\mu\sin(2(\theta_1-\theta_2))\dot\theta_1^2 - \mu\frac{g}{l_1}\sin\theta_2\cos(\theta_1-\theta_2) = 0$$

Finally solving for $\ddot\theta_1$:

$$\ddot\theta_1 = \frac{\frac{\mu}{\lambda}\dot\theta_2^2\sin(\theta_1-\theta_2) + \frac{g}{l_1}\sin\theta_1 + \frac{1}{2}\mu\sin(2(\theta_1-\theta_2))\dot\theta_1^2 - \mu\frac{g}{l_1}\sin\theta_2\cos(\theta_1-\theta_2)}{\mu\cos^2(\theta_1-\theta_2) - 1}$$

Eliminating $\ddot{\theta}_1$ and solving for $\ddot{\theta}_2$: $(1) \cdot \cos(\theta_1 - \theta_2) - (2)$ gives

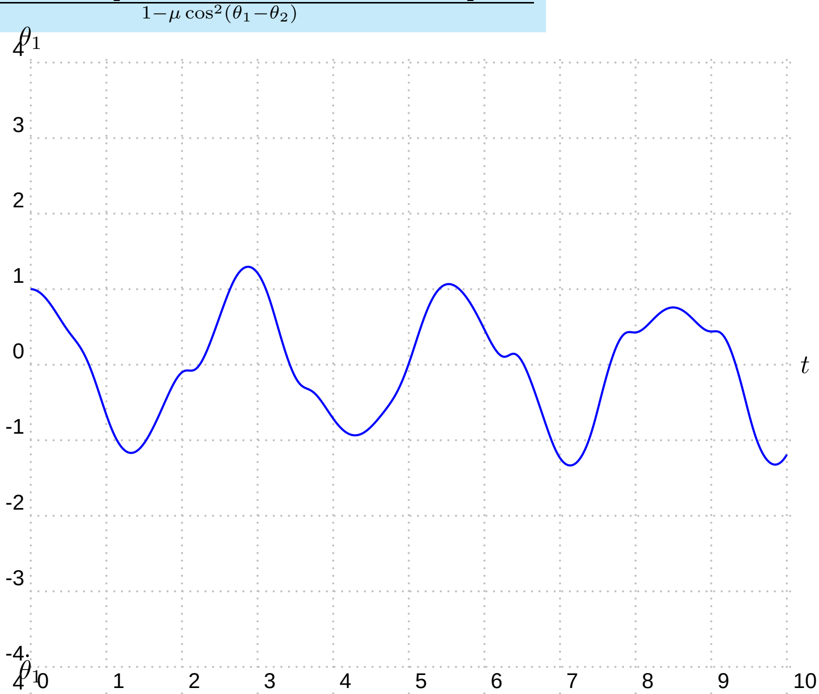
$$l_2 \left(\mu \cos^2(\theta_1 - \theta_2) - 1\right) \ddot{\theta}_2 + \frac{1}{2}\mu l_2 \sin(2(\theta_1 - \theta_2))\dot{\theta}_2^2 + g \sin\theta_1 \cos(\theta_1 - \theta_2) + l_1\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - g \sin\theta_2 = 0$$

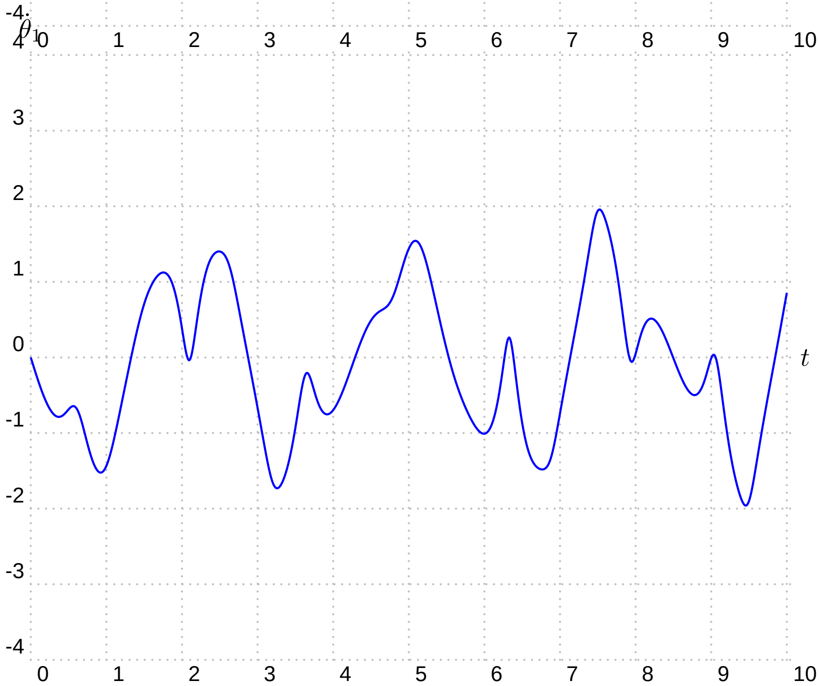
With $\lambda = \frac{l_1}{l_2}$ we simplify again

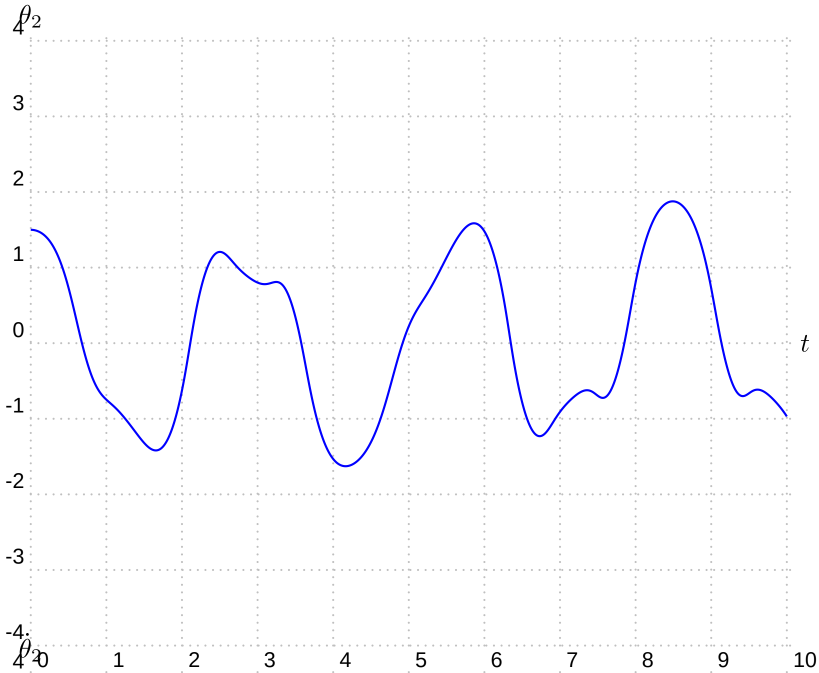
$$\left(\mu \cos^2(\theta_1 - \theta_2) - 1\right) \ddot{\theta}_2 + \frac{1}{2}\mu \sin(2(\theta_1 - \theta_2))\dot{\theta}_2^2 + \frac{g}{l_2} \sin\theta_1 \cos(\theta_1 - \theta_2) + \lambda\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - \frac{g}{l_2} \sin\theta_2 = 0$$

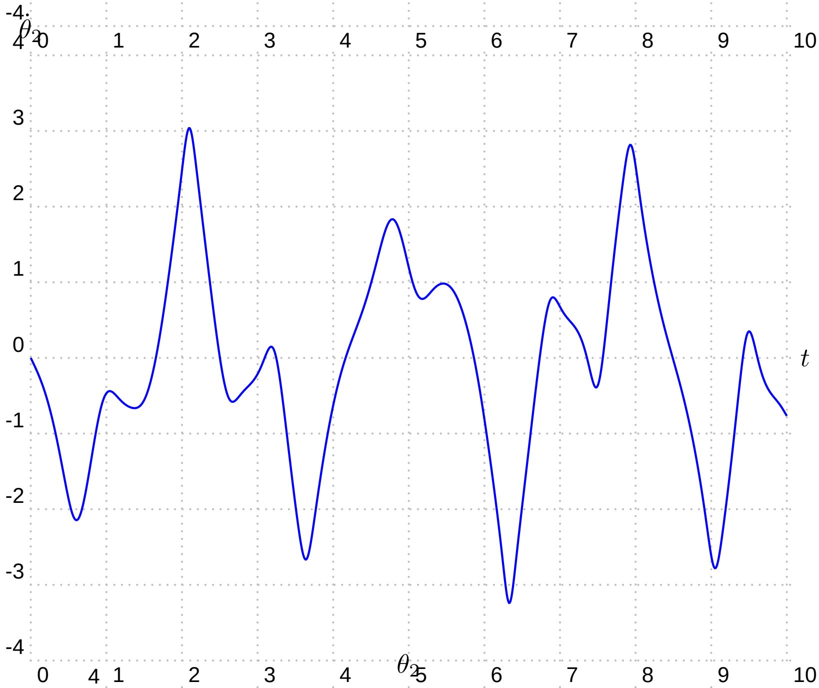
Finally solving for $\ddot{\theta}_2$:

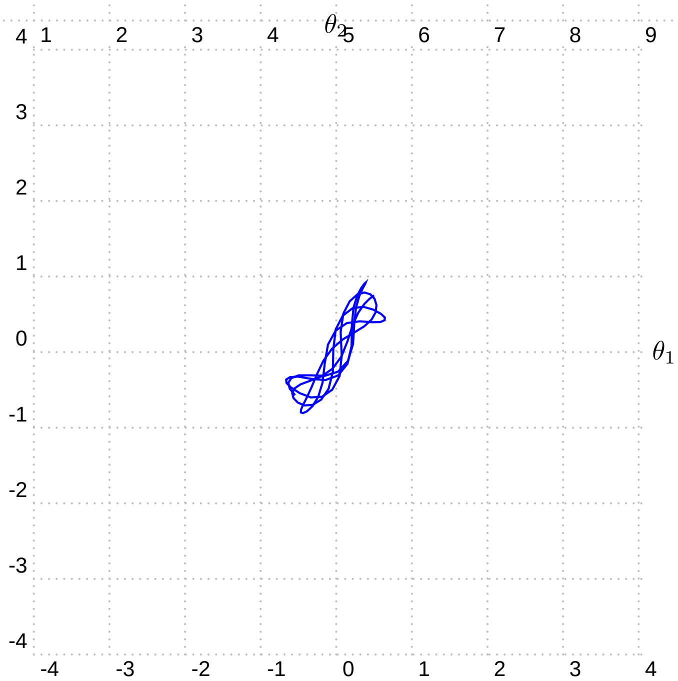
$$\ddot{\theta}_2 = \frac{\frac{1}{2}\mu \sin(2(\theta_1 - \theta_2))\dot{\theta}_2^2 + \frac{g}{l_2} \sin\theta_1 \cos(\theta_1 - \theta_2) + \lambda\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - \frac{g}{l_2} \sin\theta_2}{1 - \mu \cos^2(\theta_1 - \theta_2)}$$

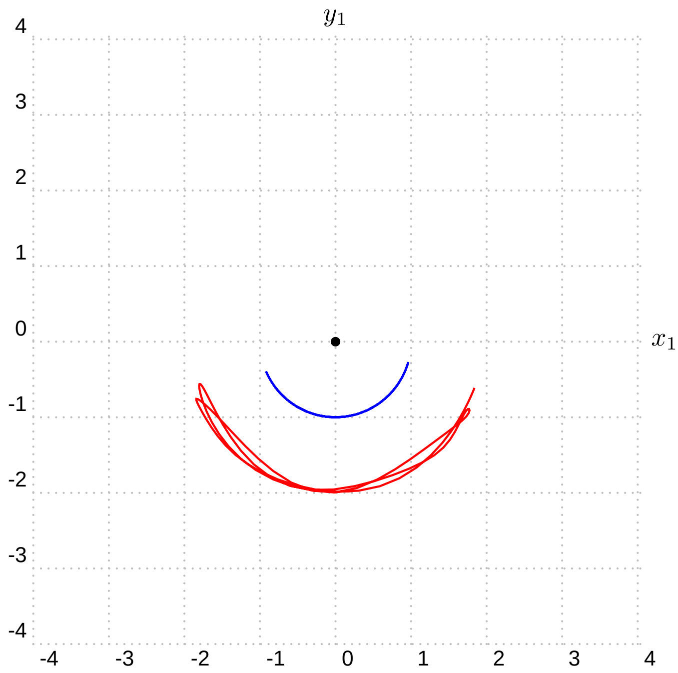





θ_2
4
3
2
1
0
-1
-2
-3
-4
0 1 2 3 4 5 6 7 8 9 10
t


$\dot{\theta}_2$
4
3
2
1
0
-1
-2
-3
-4
0 1 2 3 4 5 6 7 8 9 10
t


θ_2
4
3
2
1
0
-1
-2
-3
-4
-4 -3 -2 -1 0 1 2 3 4
θ_1



y_1
x_1
4
3
2
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3
4