The Mathematical Details for using a Helix as a Spring . . .

In Figure 1 you see a cylindrical spring (shape of a helix) with 10 windings, with an equilibrium position height h_0. The equilibrium position radius is $r_0 = \frac{d_0}{2}$.

In Figure 2 the spring is stretched – its new height is h_1 and its new radius is $r_1 = \frac{d_1}{2}$.

Questions

- What's the length of the spring?
- Now that the spring has a fixed length, what is the radius $r_1(h_0, r_0, h_1)$?

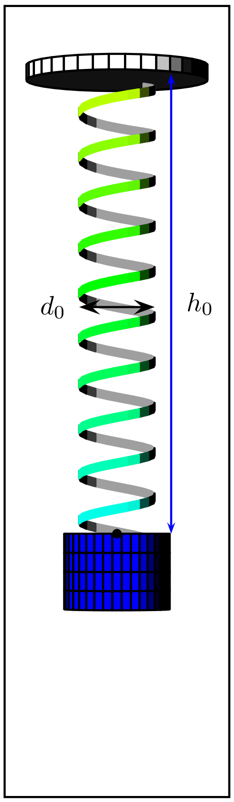


Figure 1

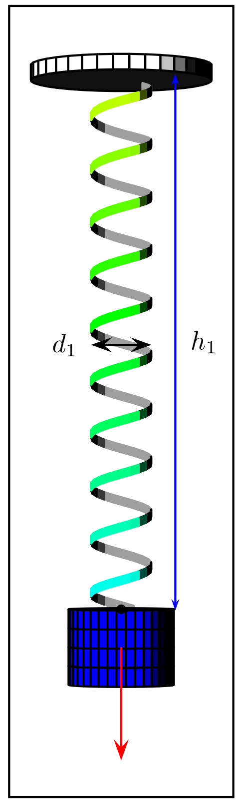


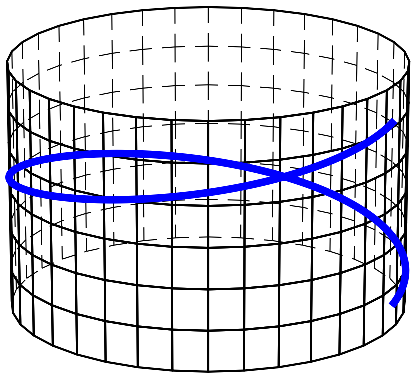
Figure 2

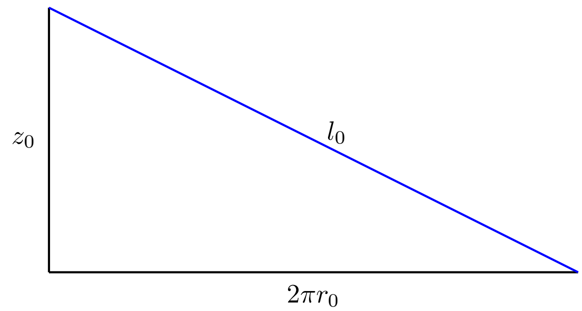
Assumptions

For simplicity, we assume that all windings are equidistant apart (the cylindrical spiral keeps the shape of a helix) and we ignore any physical properties of a 'solid' helix, e. g. thickness of the helix line, mass of the spring, material, temperature, elasticity, torsion, etc., etc., and, of course, etc.

Calculations

Let z_0 be the *height* of one single winding, so $h_0 = nz_0$ is the *total height* of a spring with n windings.





A little trick:

Roll the cylinder on the plane and trace the helix onto the plane. Thus the helix turns into a straight line. Drawing a right-angled triangle and using the Pythagorean Theorem (where the one cathetus is the height z_0, the other cathetus is the circumference $2\pi r_0$ of the cylinder and the hypotenuse is the length l_0 of the one winding) we can simply calculate the length of one winding.

The *length of one single winding* is

$$l_0 = \sqrt{z_0^2 + 4\pi^2 r_0^2} \tag{1}$$

This gives the *total length* of the spring

$$L_0 = nl_0$$

Now we stretch the spring and calculate all that for Figure 2.

Let z_1 be the *height* of one single winding, so $h_1 = nz_1$ is the *total height* of the stretched spring with n windings.

The *length of one single winding* is

$$l_1 = \sqrt{z_1^2 + 4\pi^2 r_1^2} \tag{2}$$

This gives the *total length* of the stretched spring

$$L_1 = nl_1$$

Now that this is the same spring, the length is constant: $L_0 = L_1$.

Equating (**??**) and (**??**) and solving for r_1^2, we get,

$$r_1^2 = \frac{1}{4\pi^2 n^2} h_0^2 - \frac{1}{4\pi^2 n^2} h_1^2 + r_0^2$$

Setting up the oscillation

Coordinate setup: The h-axis points downwards and the origin $h = 0$ is defined as top of the spring.

Now let the spring do a harmonic oscillation with an *amplitude* $\hat{h} < h_0$ and let the oscillation start from the *equilibrium position* h_0 downwards with an *angular frequency* $\omega = \frac{2\pi}{T}$ where T is the *period of oscillation*.

$$h_1(t) = h_0 + \hat{h} \cdot \sin(\omega t) = h_0[1 + \frac{\hat{h}}{h_0} \sin(\omega t)]$$

This gives after some basic arithmetic

$$\begin{aligned} r_1(t) &= \sqrt{\frac{1}{4\pi^2 n^2} h_0^2 \{1 - [1 + \frac{\hat{h}}{h_0} \sin(\omega t)]^2\} + r_0^2} \\ &= \sqrt{r_0^2 - \frac{\hat{h} h_0}{4\pi^2 n^2} \sin(\omega t)[2 + \frac{\hat{h}}{h_0} \sin(\omega t)]} \end{aligned}$$

Results

The more windings a spring has, the less $r_1(t)$ differs from r_0.

Now let's discuss the following two intervals of time

- $]0; \frac{T}{2}[$ – enlarging the spring $l_1(t) > l_0$

  $$\sin(\omega t) > 0 \quad \Rightarrow \quad r_1(t) < r_0$$

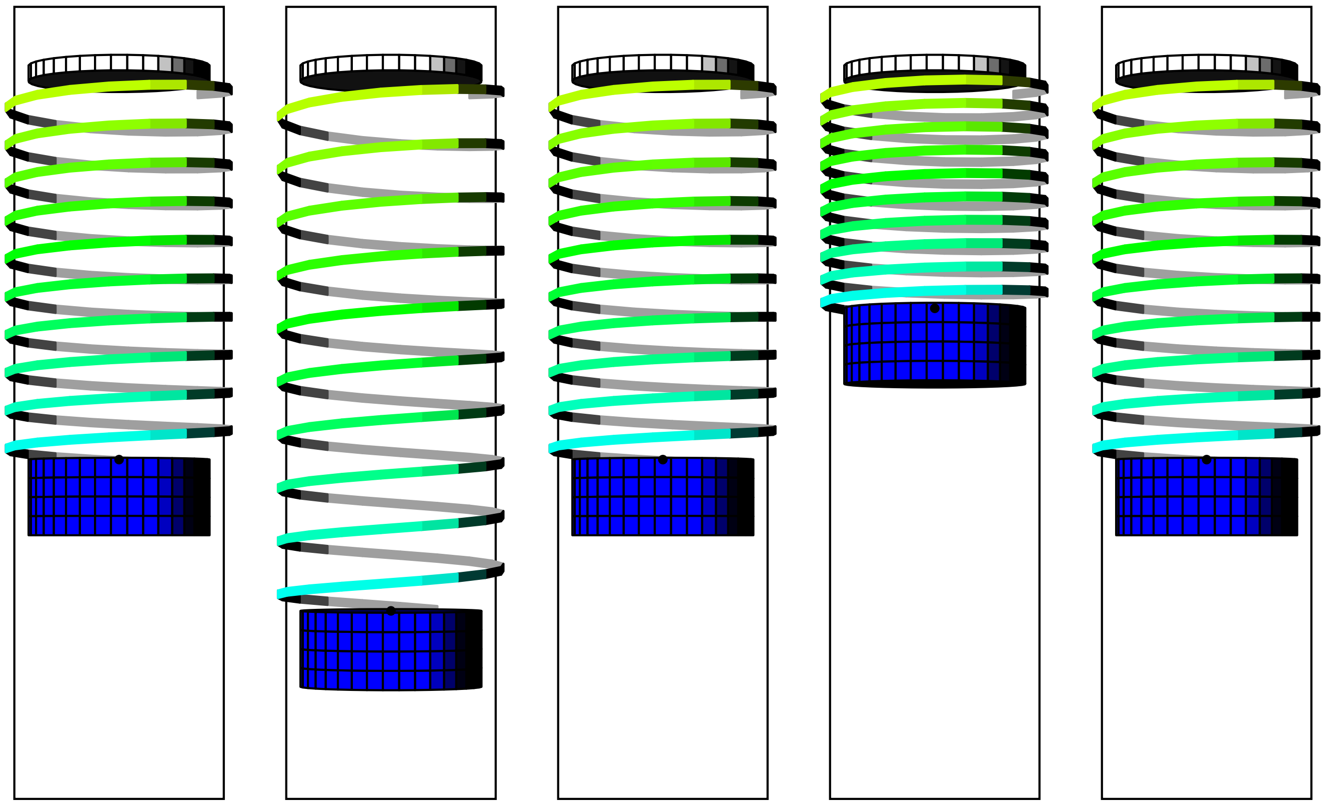
- $]\frac{T}{2}; T[$ – shortening the spring $l_1(t) < l_0$

  $$\sin(\omega t) < 0 \quad \Rightarrow \quad r_1(t) > r_0$$

Now if you calculate *by hand* some explicit examples choosing the variables h_0, r_0, $\hat{h}$, you will see that the *radius correction* is simply very tiny. Radius correction is needed, when the stretching of the spring gets large. However that disturbs the harmony of the oscillation and catapults the spring out of its Hookian limitations – these won't be the conditions for a preferred harmonic oscillation. So there is no big error to set: $r_1(t) \approx r_0$

Example

Here you see some excerpts of a harmonic oscillation ($n = 10$, $h_0 = 5$, $r_0 = 1,5$, $\hat{h} = 2$) including the *radius correction* and there is no visible change for $r_1(t)$.



$t = \dfrac{0}{360}T$ $t = \dfrac{90}{360}T$ $t = \dfrac{180}{360}T$ $t = \dfrac{270}{360}T$ $t = \dfrac{360}{360}T$

The source code

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    \pstVerb{%
      /amplitude \i\space sin 0.4 mul 1 add 5 mul def
      /radius \i\space sin 0.4 mul 1 add 2 exp neg 1 add 25 mul 4 div pi 2 exp div 100 div 1.5 2 exp add 0.5 exp def
    }
    \psframe(-1.4,-3.5)(1.4,7)
    \psSolid[object=cylindre,r=1.2,h=0.2,ngrid=1 36](0,0,6)
    \pshelices[incolor=gray!75,R=radius,h=amplitude,hue=0.2 0.5,grid,RotY=180,spires=10,dZ=0.1](0,0,6)
    \psSolid[object=cylindre,r=1.2,h=1,ngrid=4 36,fillcolor=blue](0,0,amplitude neg 5 add)
    \psPoint(0,0,amplitude neg 6 add){E1}
    \psdot(E1)
    \rput(0,-4){$t=\dfrac{\i}{360}T$}
\end{pspicture}
\qquad}
```